# FISHER MARKETS WITH LINEAR AND LEONTIEF UTILITIES

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ABSTRACT. This paper seeks to analyze the basics of equilibria in Fisher Market Games involving linear and Leontief utility functions. In particular, the analysis begins with preliminary definitions of the game at hand, then investigates a popular convex program (Eisenberg-Gale) in the linear utility setting, and finally examines the payoffs of players in the Leontief utility setting.

CONTENTS



## 1. PRELIMINARIES

We first lay out some preliminary definitions.

**Definition 1.1.** We define a "Fisher Market" as a set $N = [n]$ buyers and a set $M = [m]$ of divisible goods, such that each buyer has an initial budget $B_i > 0$ and utility function $u_i : [0,1]^m \to \mathbb{R}$ (i.e. utility per unit amount), and we assume unit supply of each good and unit total budget $\sum_{i=1}^{n} B_i = 1$.

**Definition 1.2.** We will consider utility functions belonging to the "Constant Elasticity of Substitution" family, i.e. functions of the form

$$u_i(\mathbf{x}_i) = \left( \sum_{j=1}^{m} a_{ij} x_{ij}^{\rho} \right)^{1/\rho}$$

such that $\rho \neq 0$, $\rho \leq 1$. In particular, the "Leontief" utility function is

$$u_i(\mathbf{x}_i) = \min_{j \in [m]} \{x_{ij}/a_{ij}\}$$

The "Cobb-Douglas" utility function is

$$u_i(\mathbf{x}_i) = \prod_{j \in [m]} x_{ij}^{a_{ij}}$$

Finally, the linear utility function is

$$u_i(\mathbf{x}_i) = \langle \mathbf{a}_i, \mathbf{x}_i \rangle = \sum_{j \in [m]} a_{ij} x_{ij}$$

*Date*: May 15, 2024.

where "valuation vector" $\mathbf{a}_i = (a_{ij})_{j\in[m]}$ is a parameter of the utility function.

**Definition 1.3.** We define a "market equilibrium" as a tuple $(\mathbf{p}, \mathbf{x})$ where $\mathbf{p} = (p_1, ..., p_m)$ is a vector of prices and $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n) \in \mathbb{R}^{m\times n}$ an allocation of the $m$ items such that the following holds. For all $i \in [n]$, $\mathbf{x}_i$ maximizes $u_i$ subject to the budget constraint $\langle \mathbf{x}_i, \mathbf{p}\rangle \le B_i$ (hence equality is satisfied), and for all items with $p_j > 0$ the market is cleared, i.e.

$$\sum_{i=1}^{n} x_{ij} = 1$$

**Definition 1.4.** Given a set of items $[m]$ and agents $[n]$ with budgets $B_i$, a "Fisher Market Game" is defined by the following. Each agent $i$'s strategy space consists of possible reported valuations $\mathbf{s}_i$: $S_i = \{\mathbf{s}_i \,|\, \mathbf{s}_i \in \mathbb{R}^m_{\ge 0}\}$. If $\mathbf{x}(\mathbf{s}) = (\mathbf{x}_1(\mathbf{s}), ..., \mathbf{x}_1(\mathbf{s}))$ denotes the market allocation for strategy profile $\mathbf{s}$, then for all $i \in [n]$, agent $i$ has utility $u_i(\mathbf{x}_i(\mathbf{s})) =: u_i(\mathbf{s})$ for short.

**Definition 1.5.** We say that an agent $i$ "can secure a payoff of $\alpha$" at strategy $(\mathbf{s}_i, \mathbf{s}_{-i}) \in S$ if there is an $\epsilon > 0$ and $\overline{\mathbf{s}}_i \in S_i$ such that

$$||\mathbf{s}'_{-i} - \mathbf{s}_{-i}||_2 < \epsilon \Rightarrow u_i(\overline{\mathbf{s}}_i, \mathbf{s}'_{-i}) \ge \alpha$$

## 2. Linear Utilities

**Definition 2.1.** The Eisenberg-Gale convex program for equilibrium computation is to maximize

$$\sum_{i=1}^{n} B_i \ln(u_i(\mathbf{x}_i))$$

subject to the constraints

$$x_{ij} \ge 0, \quad u_i = \left(\sum_{j=1}^{m} a_{ij} x_{ij}^{\rho}\right)^{1/\rho} \quad \forall i \in [n], \forall j \in [m]$$

$$\sum_{i=1}^{n} x_{ij} \le 1 \;\; \forall j \in [m]$$

Note that for the case of linear utility functions, the feasible set is bounded by linear inequalities and we are maximizing a summation of concave functions of linear maps, all of which preserve concavity, the objective is also concave.

**Theorem 2.2.** *The Eisenberg-Gale convex program yields a market equilibrium for linear utility-based Fisher markets.*

*Proof.* We set the Lagrange multipliers of the program to be the prices themselves. That is, our Lagrangian relaxation is the following:

$$\min_{\mathbf{p}} \max_{\mathbf{x}\ge 0} L(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{p}} \max_{\mathbf{x}\ge 0} \sum_{i=1}^{n} B_i \ln\langle \mathbf{a}_i, \mathbf{x}_i\rangle + \sum_{j=1}^{m} p_j \left(1 - \sum_{i=1}^{n} x_{ij}\right)$$

$$= \min_{\mathbf{p}} \max_{\mathbf{x}\ge 0} \sum_{i=1}^{n} B_i \ln\langle \mathbf{a}_i, \mathbf{x}_i\rangle + \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} \langle \mathbf{x}_i, \mathbf{p}\rangle$$

hence optimality is attained when

$$p_j > 0 \Rightarrow 0 = \frac{\partial L}{\partial p_j} = \sum_{i=1}^{n} x_{ij} - 1$$

and

$$x_{ij} > 0 \Rightarrow 0 = \frac{\partial L}{\partial x_{ij}} = B_i \frac{a_{ij}}{\langle \mathbf{a}_i, \mathbf{x}_i \rangle} - p_j \Rightarrow \frac{B_i}{\langle \mathbf{a}_i, \mathbf{x}_i \rangle} = \frac{p_j}{a_{ij}} \tag{2.3}$$

and where the final equality is replaced with "$\leq$" in general, since the right hand side could diverge to infinity for $x_{ij} = 0$ (agent $i$'s valuation on item $j$ could be close to zero). Thus, for $x_{ij} > 0$ and some different item $j' \in [m]$, we have

$$\frac{p_j}{a_{ij}} = \frac{B_i}{\langle \mathbf{a}_i, \mathbf{x}_i \rangle} = \frac{B_i}{u_i(\mathbf{x}_i)} \leq \frac{p_{j'}}{a_{ij'}} \iff \frac{a_{ij'}}{p_{j'}} \leq \frac{a_{ij}}{p_j}$$

That is, the utility per price of any item that agent $i$ is allocated is at least as good as that of any other item (all agents only purchase optimal items). Finally, by (1.13) we may sum over the items $j \in [m]$ as

$$\langle \mathbf{x}_i, \mathbf{p} \rangle = \sum_{j=1}^{m} x_{ij} p_j = \sum_{j=1}^{m} x_{ij} \frac{B_i a_{ij}}{\langle \mathbf{a}_i, \mathbf{x}_i \rangle} = B_i$$

so that an equilibrium exists in the Fisher market where participants only purchase optimal items and spend the entirety of their budgets $B_i$, as the feasible set is non-empty and the objective is concave. □

**Remark 2.4.** Note that the solution above is Pareto optimal since any alternative allocation has a lesser $\sum_{i=1}^{n} B_i \ln\langle \mathbf{a}_i, \mathbf{x}_i \rangle$ so that at least one of the terms in the summation has decreased, hence at least one player is worse off.

**Corollary 2.5.** *A Fisher market game with linear utilities has a symmetric, pure Nash equilibrium where the payoffs are identical to the payoffs obtained from truthful play.*

**Theorem 2.6.** *The Fisher market game with linear utilities has a Price of Anarchy that is $O(n)$.*

*Proof.* We normalize the valuation vectors so that $\sum_{j=1}^{m} a_{ij} = \delta$ for all $i \in [n]$. Then, if each agent $i$ assigns value only to its optimal item, the agent can guarantee payoff

$$u_i \geq \frac{B_i a_i^{\max}}{\sum_{k=1}^{n} B_k} = \frac{a_i^{\max}}{n}$$

We then have that the Price of Anarchy is

$$\mathrm{PoA} = \frac{\max_{s \in S} \mathrm{welf}(s)}{\min_{s \in Eq.} \mathrm{welf}(s)} \leq \frac{\sum_{i=1}^{n} B_i a_i^{\max}}{\frac{1}{n} \sum_{i=1}^{n} B_i a_i^{\max}} = n$$

□

**Remark 2.7.** Please note that in [2] the "PoA" refers to 1/PoA as we have defined it here. This caused a great deal of trouble for the author.

**Theorem 2.8.** *The Fisher market game with linear utilities has a Price of Anarchy that is $\Omega(\sqrt{n})$.*

*Proof.* Consider a set of $n = m^2 + m$ agents and $m$ items such that $B_i = 1$ for all $i \in [m^2 + m]$. For $i \in [m]$, define valuation vectors $\mathbf{a}_i = (0, ..., 0, 1, 0, ..., 0)$ where the 1 is in the $i$th spot, and for $i \in \{m+1, ..., m^2 + m\}$ define valuation vectors $\mathbf{a}_i = (1/m, ..., 1/m)$. The optimality conditions of the EG program then guarantee that, for $i \in [m]$

$$\frac{B_i}{\langle \mathbf{a}_i, \mathbf{x}_i \rangle} = \frac{p_i}{a_{ii}} \Rightarrow \frac{1}{x_{ii}} = \frac{p_i}{1} \Rightarrow \frac{1}{p_i} = x_{ii}$$

and for $i \in \{m+1, ..., m^2 + m\}$,

$$\frac{1}{\sum_{\ell=1}^m a_{i\ell} x_{i\ell}} = \frac{p_j}{a_{ij}} \Rightarrow \frac{m}{\sum_{\ell=1}^m x_{i\ell}} = mp_j \Rightarrow \sum_{\ell=1}^m x_{i\ell} = \frac{1}{p_j}$$

so that all $p_j =: p$ must be the same. Then, the last EG optimality condition is

$$1 = \sum_{i=1}^n x_{ij} \Rightarrow \frac{m^2 + m}{p} = \sum_{i=1}^{m+m^2} \sum_{j=1}^m x_{ij} = \sum_{j=1}^m 1 = m$$

$$\Rightarrow p_j = p = m + 1$$

and so $x_{ii} = \frac{1}{m+1}$ for all $i \in [m]$, $x_{ij} = \frac{1}{m^2+m}$ for $i \in \{m+1, ..., m^2 + m\}$, and all other $x_{ij}$ are zero. Thus, a market equilibrium under truthful valuations would yield social welfare $\frac{m}{m+1} + \frac{m^2}{m(m+1)} = \frac{2m}{m+1}$. Since giving unit (total) amount of item $i$ to agent $i$ for $i \in [m-1]$ yields social welfare at least $m - 1$, the optimal social welfare is at least $m - 1$. By Corollary 1.15 there exists a Nash equilibrium with social welfare $2m/(m + 1)$, hence the Price of Anarchy is

$$1/\text{PoA} \leq \frac{2m}{(m-1)(m+1)}$$

in a scenario with a maximum welfare at equilibrium that is equal to its minimum welfare at equilibrium. We have then PoA $\geq \frac{(m+1)(m-2)}{2m}$ so that the quantity is $\Omega(m)$, or $\Omega(\sqrt{n})$. □

## 3. Leontief Utilities

**Lemma 3.1.** *If a pair of strategies* $(\mathbf{s}_1, \mathbf{s}_2)$ *is a pure Nash equilibrium of the Fisher market game with two agents and Leontief utilities, the utilities satisfy*

$$u_i(\mathbf{s}_1, \mathbf{s}_2) \leq B_i / a_i^{max}$$

**Lemma 3.2.** *In a two-player Leontief-based Fisher market game, the uniform strategies ensure player 1 gets* $u_1 \geq B_1 / a_1^{max}$.

**Theorem 3.3.** *Uniform strategies are an equilibria for the two-player Fisher market game with Leontief utilities, and the utilities are*

$$u_1 = \frac{B_1}{\max_{j \in [m]} a_{1j}}, \quad u_2 = \frac{B_2}{\max_{j \in [m]} a_{2j}}$$

*Proof.* For every agent $i \in [n]$ we define

$$a_i^{\max} := \max_{j \in [m]} a_{ij}, \quad a_i^{\min} := \min_{j \in [m]} a_{ij}$$

$$s_i^{\max} := \max_{j \in [m]} s_{ij}, \quad s_i^{\min} := \min_{j \in [m]} s_{ij}$$

If we fix the strategy $\mathbf{s}_2$ of agent 2, then the best response of agent 1 would be to request exactly what is available, i.e.

$$\mathbf{s}_1 = (s_{1j})_{j \in [m]}, \quad s_{1j} = 1 - \frac{s_{2j}}{s_2^{\max}} B_2$$

which yields utility

$$u_1(\mathbf{s}_1, \mathbf{s}_2) = \min_{j \in [m]} \frac{1 - \frac{s_{2j}}{s_2^{\max}} B_2}{a_{1j}}$$

and where, for all items $j \in [m]$, agent 2 is allocated

$$x_{2j} = \frac{s_{2j}}{s_2^{\max}} B_2$$

with utility

$$u_2'(\mathbf{s}_1, \mathbf{s}_2) = B_2 / s_2^{\max}$$

Then, since $\sum_{j=1}^m p_j = \sum_{i=1}^n B_i = 1$,

$$u_i'(\mathbf{s}_1, \mathbf{s}_2) = \frac{B_i}{\sum_{j=1}^m p_j s_{ij}}, \quad \frac{B_i}{s_i^{\max}} \leq u_i'(\mathbf{s}_1, \mathbf{s}_2) \leq \frac{B_i}{s_i^{\min}} \tag{3.4}$$

Then, if either agent does not recieve utility $u_i'(\mathbf{s}_1, \mathbf{s}_2) = B_i / s_i^{\max}$, one of the agents increases their allocation by following the best response strategy above. Thus, by Lemma 1.14, 1.15 and equation 1.17, the uniform strategies yield a Nash equilibrium with $u_i = B_i / a_i^{\max}$. □

**Theorem 3.5.** *The Fisher market game with Leontief utilities has a Price of Anarchy that is $O(n)$.*

*Proof.* We first note that optimizing allocation in Leontief utility-based Fisher markets involves $x_{ij}/a_{ij}$ equal across all goods $j$ for a specific agent $i$ (else there is allocation that is going to waste, since equal allocation per requirement yields the same utility). Thus,

$$x_{ij} = a_{ij} u_i \Rightarrow B_i = \sum_{j=1}^m x_{ij} p_j = u_i \sum_{j=1}^m a_{ij} p_j$$

We then have

$$u_i = \frac{B_i}{\sum_{j=1}^m p_j a_{ij}} \geq \frac{B_i}{\sum_{j=1}^m p_j a_i^{\max}} = \frac{B_i}{\sum_{k=1}^n B_k a_i^{\max}}$$

since there is unit supply of every item and an equivalent net budget. Moreover, each player gets at most unit amount of any item hence social welfare is bounded as

$$\sum_{i=1}^n u_i(\mathbf{x}_i) \leq \sum_{i=1}^n u_i(\mathbf{1}) = \sum_{i=1}^n \frac{1}{a_i^{\max}}$$

Then, the Price of Anarchy is at most

$$\frac{\sum_{\ell=1}^n 1/a_\ell^{\max}}{\sum_{i=1}^n (B_i / \sum_{k=1}^n B_k a_i^{\max})}$$

so that under the assumption of delta normalization, i.e $a_i^{\max} B_i = \delta$, we have that

$$\text{PoA} \leq \frac{(1/\delta) \sum_{\ell=1}^n B_\ell}{\sum_{i=1}^n B_i / (\sum_{k=1}^n B_k (\delta / B_i))} = \frac{(\sum_{k=1}^n B_k)^2}{\sum_{i=1}^n B_i^2}$$

so that by Holder's Inequality,

$$= n\frac{(\sum_{k=1}^n B_k)^2}{(\sum_{i=1}^n B_i^2)(\sum_{i=1}^n 1^2)} \leq n\frac{(\sum_{k=1}^n B_k)^2}{(\sum_{i=1}^n B_i \cdot 1)^2} = n$$

Hence, the Price of Anarchy is $O(n)$. $\square$

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